ISRAEL JOURNAL OF MATHEMATICS 86 (1994), 373–396

SPATIAL POLYHEDRA WITHOUT DIAGONALS

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ABSTRACT

We provide new polyhedra without diagonals, and we discuss their embeddings in euclidean 3-space with maximal symmetries starting with a complete classification of their combinatorial properties : orientable neighborly 2-pseudomanifolds with 9 vertices or Mendelsohn triple systems $S_2(2,3,9)$. This article was motivated by the longstanding and still open question: find a triangulated 2-manifold which can not be embedded in 3-space. Furthermore, we applied tested and improved algorithms for realizing oriented matroids or finding final polynomials.

Received March 2, 1992 and in revised form December 6, 1992

1. Introduction

By **a polyhedron without diagonals** or also **neighborly polyhedron** we mean a polyhedron P in euclidean 3-space R^3, in which for every two vertices x, y of P, the line-segment $[x, y]$ joining x and y is an edge of P. In particular it follows that bd$P$, the boundary complex of P, is a simplicial 2-complex and that the graph of P (whose vertices and edges are the vertices and edges of P, resp.) is a complete graph.

As far as we know, the only known neighborly polyhedra in R^3 are the tetrahedron (the only convex one) and Császár's torus with 7 vertices ([12], see also [16], [1], [9]). In this article we present several new neighborly polyhedra, each of them with 9 vertices. The boundaries of these polyhedra, however, are not 2-manifolds but rather *pseudomanifolds* or *pinched manifolds.* The requirement is that each edge occurs in exactly two triangles.

From the combinatorial point of view, the boundary structures of our polyhedra are *simple block designs* $S_2(2,3,9)$ which have been classified in the literature ([19], [20], [17]). In Section 5 and 6, we give various polyhedral embeddings into R^3. Our investigation is based on a method using *oriented matroids* described in Section 3, and it requires extensive computer calculations.

It is well known that the next case of a neighborly 2-manifold (after Császár's torus) is the case of genus 6 with 12 vertices, see [1], [5], [21]. In a yet unpublished result, the first author has found 59 distinct orientable types. However, the embedding problem is open for quite a long time and we hope that this problem can be tackled by the methods of this article.

1.1 PROBLEM: *Does there exist a neighborly polyhedron with 12 vertices whose boundary is a 2-manifold ?*

2. Neighborly Pseudomanifolds and Block Designs

A **block design** $S_2(2,3,n)$ is a collection of abstract triples (or blocks) of elements $1, 2, \dots, n$ s.t. every pair of elements is contained in exactly two triples. It is called a **simple block design** if each triple occurs at most once, i.e. if there are no repeated blocks. Interpreting the triples as triangles of a simplicial 2-complex M with n vertices we have :

1. any two vertices $x, y \in M$ are joined by an edge of M,
2. every edge is contained in exactly two triangles.

This implies that the link of a vertex is a union of closed cycles. We call such a complex a **neighborly 2-pseudomanifold**, briefly NPM. If the link of a vertex is one single cycle we call the vertex **regular**, otherwise **singular**. If the cycles in the link of a singular vertex x are of lengths $n_1, n_2, \ldots, n_t$, x is **singular of type** $(n_1, n_2, \ldots, n_t)$, and of **multiplicity** t. (Clearly $n_i \geq 3$ for each i.) A 2-pseudomanifold all of whose vertices are regular is called a **2-manifold**.

For a general block design $S_\lambda(2,3,n)$, every pair of elements is contained in exactly λ triples. Especially, a block design $S_1(2,3,n)$ is called a **Steiner triple system** (STS). An oriented simple $S_2(2,3,n)$ (the blocks are cyclic oriented triples and every oriented pair of elements is contained in a unique block) is also known as **Mendelsohn triple system**. Its topological counterpart is an orientable NPM.

A **cyclic block design** $S_2(2,3,n)$ is a design with pointset Z_n (= additive group of integers modulo n) such that any block $\{a_1, a_2, a_3\}$ implies $\{a_1+1, a_2+1, a_3+1\}$ to be a block as well. A **cycle class** is a set of blocks $\{\{a_1+i, a_2+i, a_3+i\} \mid i \in Z_n\}$, where $\{a_1, a_2, a_3\}$ is a block. With this definition we call $S_2(2,3,n)$ a **1-rotational block design** if the pointset is $Z_{n-1} \cup \{\infty\}$, where $\infty + 1 = \infty$, see [18].

If a 2-complex M is a neighborly pseudomanifold NPM and x is a singular vertex of M of type $(n_1, n_2, \ldots, n_t)$, then **splitting** x is the operation of replacing x by t new vertices $x_1, x_2, \ldots, x_t$, joining each x_i by edges to the proper n_i vertices $y_{i_1}, y_{i_2}, \ldots, y_{i_{n_i}}$ which form a cycle of length n_i in $\text{link}(x, M)$, and adding the triangles $x_i y_{i_1} y_{i_{n_i}}, x_i y_{i_j} y_{i_{j+1}}$ $(1 \leq j < n_i)$, thus yielding from M a 2-pseudomanifold M', not neighborly any more, with $t-1$ vertices more than M, such that $\text{link}(x_i, M')$ is a cycle of length n_i. If we split several singular vertices of M, then the final result is independent of the order of the splitting.

A neighborly pseudomanifold NPM M is **strongly connected** if for every two triangles Δ, Δ' in M, there is a sequence $\Delta = \Delta_1, \Delta_2, \ldots, \Delta_t = \Delta'$ of triangles in M such that for each $1 \leq i < t$, $\Delta_i$and Δ_{i+1} share a common edge.

An NPM M is not necessarily strongly connected. But if it were, then splitting all its singular vertices yields a 2-manifold. If, however, M is not strongly connected but has, say, s components with respect to strong connectivity, then splitting all the singular vertices of M yields s disjoint 2-manifolds.

Since the boundary complex of every 3-polyhedron is orientable, *we assume all the 2-manifolds and NPM's in this article to be orientable.* (Note that if a pseudomanifold M_1 is obtained from an NPM M_2 by splitting a singular vertex, then M_1 is orientable if and only if M_2 is.)

Let M be an NPM with n vertices, then it has $n(n-1)/2$ edges and $n(n-1)/3$ triangles. Hence $n \not\equiv 2 \bmod 3$. The link of a regular vertex of M is a cycle of length $n-1$. If x is a singular vertex of type $(n_1, n_2, \ldots, n_t)$, then $n_1+n_2+\cdots+n_t = n-1$. Hence there is no NPM with singular vertex and $n \leq 6$. It is also easy to check that the only NPM with $n = 7$ vertices is the Möbius torus, see [2]. Thus $n = 9$ is the first interesting case. It is well known that the smallest $n > 7$ for which there exist neighborly 2-manifolds with n vertices is $n = 12$, see e.g. [1]. Thus the most we can expect for $7 < n < 12$ is orientable NPM's with 9 and/or 10 vertices having at least one singular vertex. From now on we concentrate on NPM's with 9 vertices.

This implies: an NPM has 9 vertices, 36 edges, and 24 triangles. A singular vertex in an NPM M is necessarily of type $(3,5)$ or of type $(4,4)$. In any case its multiplicity is 2. Therefore if there are altogether p singular vertices in M, and M is strongly connected, then splitting all the singular vertices of M yields a 2-manifold M' of genus γ, say, and Euler's equation reads $9+p-36+24 = 2-2\gamma$. Hence p is odd, $\gamma = 5 - p/2$ and, as $\gamma \geq 0$, $p \leq 5$. M can be obtained from M' by *pinching* M' at certain p pairs of vertices. Thus we can say that the NPM M is a **pinched 2-manifold** of genus γ, eg if $p = 3$ then M is a pinched torus.

In [20], [19], [17] we find lists of all $S_2(2,3,9)$ block designs. From those lists (and from our independent one as well) we get :

2.1 THEOREM: *There are altogether five orientable neighborly 2-pseudomanifolds with 9 vertices.*

It is interesting to note that these five NPM's are distinguished already by the types of their singular vertices. If we denote by NPMij an NPM with i singular vertices of type $(3,5)$ and j singular vertices of type $(4,4)$, then the five NPM's are symbolized by NPM21, NPM03, NPM30, NPM01, and NPM45. The first three are pinched tori, the fourth is a pinched manifold of genus 2, and the last one is not a pinched manifold. Splitting all its 9 singular vertices yields three 2-spheres. The detailed description of the five NPM's is in Section 4. In Section 5 we will prove the main result of our article.

2.2 THEOREM: *Each of the five orientable neighborly 2-pseudomanifolds with nine vertices is geometrically embeddable in R^3.*

Each such embedding yields a neighborly 3-polyhedron with 9 vertices. However, two different geometrical embeddings of the same NPM may yield two polyhedra which differ essentially. In [9] all realizations of Möbius torus with 7 vertices were found. We are not going to carry the present investigation that far. Even the meaning of different types will be restricted here more than in [9].

Consider a triangulated torus T, and let x, y be two vertices in T far apart from each other. A pinched torus can be obtained from T by glueing (ie identifying) x and y either "inside" the torus (Figure 1a) or "outside" it (Figure 1b). In the first case we say the identification of x, y (or the realization of the torus at the singular vertex $x = y$) is *of type I* (In), in the other case it is of type O (Out). If we consider the unfolded torus as a rectangle in the usual way, and we think of the rectangle as two-sided, where the side seen in front being the outside of the torus, then the type I identification can be thought of as the points x, y lying on the unseen side of the rectangle (Figure 1c), while in the type O identification, they lie in front (Figure 1d). This notion can be generalized easily to pinched manifolds of higher genus.

Two realizations of a strongly connected NPM M will be considered **type-different** if M has a singular vertex which in one realization of M is of type

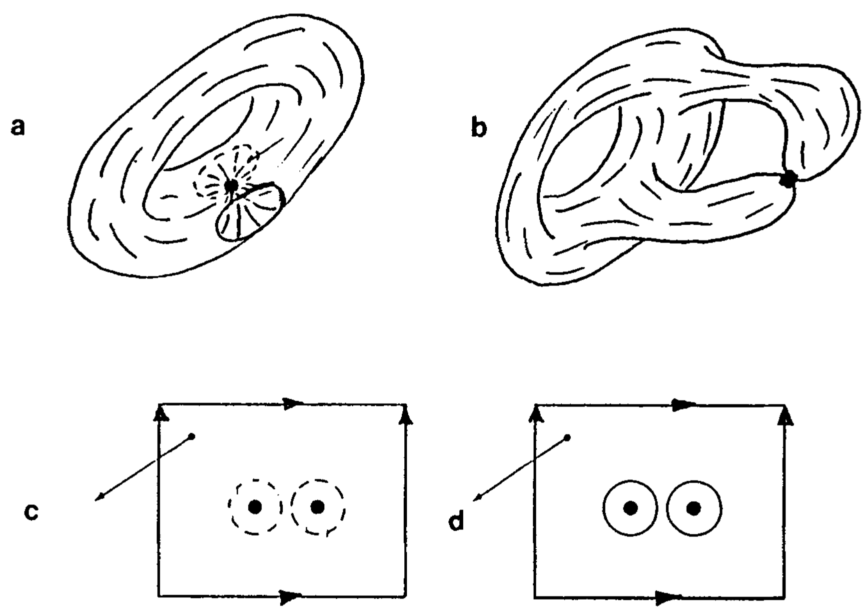

Figure 1: Two types of identification of two vertices

I while in the other it is of type O. Thus a pinched manifold with 3 singular vertices may have up to $2^3 = 8$ different topological realizations, yielding at most 8 different neighborly polyhedra.

The neighborly polyhedra in this article are looked for as follows. For a given NPM M, we pick a symmetry of M of the highest degree, such that M can be geometrically embedded in R^3 with this symmetry. Next we look for all type-different embeddings in R^3 having that symmetry.

3. Oriented Matroids

This article deals with orientable neighborly pseudomanifolds and their realizations in 3-dimensional euclidean space. As a necessary condition for a symmetric realization of a neighborly pseudomanifold in the 3-dimensional affine euclidean space, we require the existence of a symmetric **neighborly matroid pseudomanifold** defined as a pair of the NPM together with an admissible oriented matroid. We are going to define this in the following.

Among the possible ways of defining an oriented matroid χ, we use a definition involving hyperline sequences, see e.g. [8].

We write the finite set of n points as $E = \{1, \ldots, n\}$, and we use $\Lambda(n,d)$ for the set of all tuples $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_d) \in E^d$, $1 \leq \lambda_1 < \lambda_2 < \cdots < \lambda_d \leq n$. To each **oriented hyperline** $l = (l_1, \ldots, l_{d-2}) \in \Lambda(n, d-2)$, we assign a **normalized hyperline sequence**

$$h_l = (l_1, \ldots, l_{d-2} \,|\, s^l_{d-1} l_{d-1}, \ldots, s^l_n l_n)$$

with $\{l_1, \ldots, l_n\} = E$, $l_{d-1} < l_d, \ldots, l_n$, and $s^l_i \in \{-1, 1\}$, $d-1 \leq i \leq n$, ($s^l_{d-1} = 1$); and we assign corresponding **bracket signs** (= signs of formal $d \times d$-subdeterminants)

$$\mathrm{sign}[l_1, \ldots, l_{d-2}, s^l_j l_j, s^l_k l_k] := 1$$

for all j, k: $d-1 \leq j < k \leq n$, such that for all **bases** $b = (b_1, \ldots, b_d) \in \Lambda(n,d)$ and for all permutations π: $\{1, \ldots, d\} \to \{1, \ldots, d\}$

$$\mathrm{sign}[b_1, \ldots, b_d] = \mathrm{sign}(\pi)\mathrm{sign}[b_{\pi(1)}, \ldots, b_{\pi(d)}],$$

$$-\mathrm{sign}[b_1, \ldots, b_j, \ldots, b_d] = \mathrm{sign}[b_1, \ldots, -b_j, \ldots, b_d].$$

The last conditions tell us that the $\binom{d}{2}$ definitions of signs for a fixed bracket $b = [b_1, \ldots, b_d]$ with $(b_1, \ldots, b_d) \in \Lambda(n,d)$ have to be compatible. It is known that the hyperline sequences are determined by the bracket signs and vice versa.

3.1 Definition: Oriented Matroid in Terms of Hyperline Sequences. The set E together with a set of hyperline sequences with compatible bracket signs is a **uniform oriented matroid** $\chi(n,d)$ **of** n **points in rank** d.

Since we deal with 3-dimensional polyhedra, the **rank** d is 4, and we write $\chi(n,4) = \chi(n)$. Given a group G of permutations on E, we can have automorphisms of NPM's as well as automorphisms of oriented matroids defined in a straight-foreward way.

3.2 Definition: Neighborly Matroid Pseudomanifold. A **neighborly matroid pseudomanifold with** n **points and symmetry group** G, is a pair (NPM, χ) of a neighborly pseudomanifold NPM with symmetry group G and a uniform oriented matroid χ with the same set E of points and the same symmetry group G, such that χ is **compatible** to the list of triangles of the neighborly pseudomanifold NPM. *Compatible* here says that the restriction of χ to any set of 5 points formed as the union of a triangle and an edge of the NPM can be realized as a 5 point set in R^3 such that the realized edge does not intersect the realized triangle.

The notion of a neighborly matroid pseudomanifold is similar to the notion of a matroid polytope, see [6]. It is a combinatorial object midway between the abstract (or topological) NPM and its final realization.

We define a **tetrahedral partition** $TP(\mathrm{NPM},\chi)$ of a neighborly matroid pseudomanifold (NPM,χ) to be a set of bases $\{t_1,t_2,\ldots,t_k\}$, of the oriented matroid χ, with $t_i \in \Lambda(n,4)$, such that each triangle formed by deleting one element in the set of points of any base occurs either precisely twice or once, and if it occurs once, then it is a triangle of NPM. Furthermore, each triangle of an NPM occurs once, and χ is compatible, in the sense of Definition 3.2, to the list of all the above triangles.

On the topological level, a tetrahedral partition of a 2-pseudomanifold M is a 3-pseudomanifold with M as its boundary and with no interior vertices.

4. Combinatorial Description of the Five NPM's

In the sequel we describe the five NPM's, see [20], [19], [17]. In each case we give the list of triangles in the NPM. Moreover, we describe the automorphism group G of the abstract complex, and we provide neighborly matroid pseudomanifolds (NPM,χ) having a subgroup G' of G as large as possible. The notion **type-different** carries over to the oriented matroid setting. For a neighborly matroid pseudomanifold which has a tetrahedral partition, the topological type (= type I or O of the embedding) is easily determined from the partition.

We looked for type-different neighborly matroid pseudomanifolds with the symmetry group G' for all possible topological types of NPMs.

For many but not all topological types of all NPM's for which there was a corresponding neighborly matroid pseudomanifold, there was also a corresponding tetrahedral partition. We describe these tetrahedral partitions in the sequel. Apart from their abstract relevance, they are of use in constructing and understanding the neighborly polyhedra.

4.1 NPM21: A Pinched Torus.

The triangles of NPM21 are

123	129	134	146	156	157	178	189	239	247	256	257
248	268	347	358	359	367	368	458	459	469	679	789

This orientable neighborly pseudomanifold is easily seen to be a pinched torus, singular at the vertices $2, 5$ and 8. The vertices 2 and 8 are of type $(3, 5)$, the vertex 5 is of type $(4, 4)$, (Figure 2). This complex has the symmetry group of order 2 generated by $\pi = (19)(28)(37)(46)(5)$. Since this symmetry interchanges the two singular vertices of type (3,5), we can expect at most 4 topological types of neighborly polyhedra.

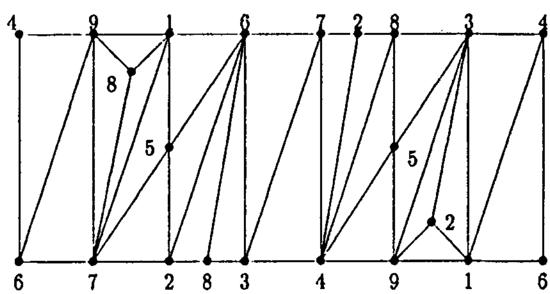

Figure 2: NPM21, a pinched torus

In looking at Figure 2, we see that after splitting the singular vertices one can get the Möbius' torus with seven vertices by shrinking the proper edges.

There are altogether 5 oriented matroids admissible to NPM21 with the symmetry π. For the method of finding them compare eg [11] or [7].

One oriented matroid is given implicitly by matrix No.1 in Section 6. The remaining four oriented matroids differ only by the sign of some brackets. The first and the second oriented matroid differ by the sign of the brackets [1348] and [2679], the second and the third by [1346] and [4679], the third and the fourth by [1345] and [5679], and the fourth and the last by [1347] and [3679], respectively.

All these five neighborly matroid pseudomanifolds have the same tetrahedral partition, namely

−1789	−1679	+1567	−3567	+3568	−2568
	−1469		$\vert$	5	$\vert$
−1239	−1349	+3459	−3457	+2457	−2458

The signs of the tetrahedra (signs of bases of the oriented matroid) provide us with useful information. This implies that we can expect at most one realized topological type.

4.2 NPM03: A Pinched Torus.

The triangles of NPM03 are

124	135	168	179	236	259	278	347	389	458	469	567
127	139	146	158	238	249	256	345	367	478	579	689

It is easily seen to be a pinched torus with three singular vertices 4,5, and 6 each of type $(4,4)$. The automorphism group of this complex is the symmetric group S_3 of order 6 generated by $\sigma = (19)(28)(37)(46)(5)$ and $\pi = (123)(465)(789)$. There are also interesting properties in the block designs terminology. The first row of the above triangles is a Steiner triple system $S_1(2,3,9)$ described by the following model of AG(2,3): $\begin{matrix}5&6&7\\9&8&3\\2&1&4\end{matrix}$, and the second row is the STS disjoint to the first: $\begin{matrix}5&7&9\\8&3&2\\1&6&4\end{matrix}$. In Figure 3 the disjoint STS are marked. In [14] is a list of a maximal number of pairwise disjoint STS with 9 vertices and the second and sixth STS form together without permutations a block design $S_2(2,3,9)$. NPM03 is the only 9NPM which can be build by two disjoint STS. The image under σ of the first STS is the second STS, and π is a symmetry to itself. In Figure 3, σ is the reflection at the centre and π is a translation.

There is no oriented matroid with the full symmetry of order 6.

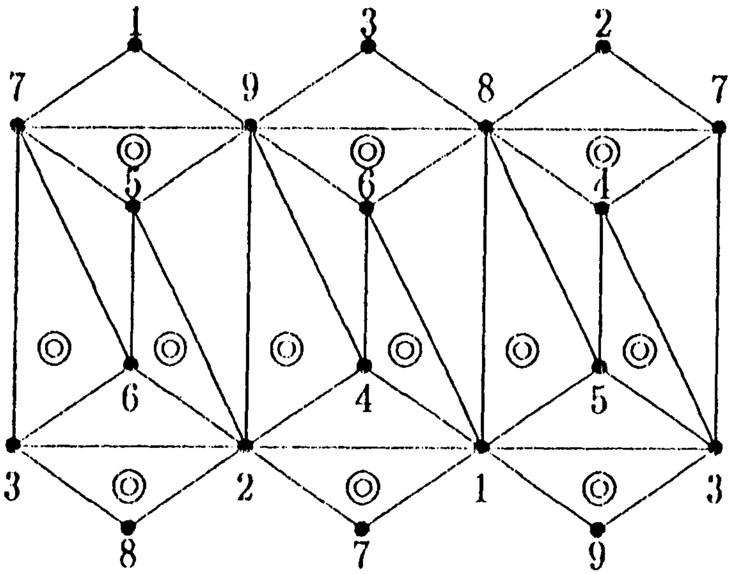

Figure 3: NPM03, a pinched torus

There exist (up to symmetries) altogether 3 oriented matroids forming together with NPM03 a neighborly matroid pseudomanifolds with the symmetry of order 3. We give the signs of all bases of the corresponding oriented matroid ordered canonically ([1234] = +, [1235] = +, ..., [1239] = +, [1245] = −, ..., [6789] = −).

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++++++-----+-------++-+++++++++++--+----++-+--++++--+--+++---+-
---------++++++++++--+++++++--+------+-----+--++++++--------+--

++++++-----+--+----++-+++++++++-+--+----++-+--++++--+--++++--+-
---------++++++++++--+++++++--+------+-----+--++++++--------+--

++++++-----+--+--+-++-++-++++++-+--+----++-+--++++--+--++++--++
---------++++++++++--+++++++--+------+-----+--++++++--------+--
```

Here too, all these three neighborly matroid pseudomanifolds share the same tetrahedral partition, see Case *OOO* later. As we shall see later in Section 5, none of these oriented matroids with order 3 symmetry is realizable.

For the symmetry σ, it turned out that there are many neighborly matroid pseudomanifolds. As σ interchanges two singular vertices, we can expect at most 4 topological types of realizations. Thus we looked for 4 distinct oriented matroids with symmetry σ which may yield these 4 types.

We found 4 such neighborly matroid pseudomanifolds with symmetry σ.

They led us to the following 4 tetrahedral partitions of NPM03. The second case is of interest later, its vertex 4 is of type O, 5 is of type I, 6 is of type O, so we denote it Case OIO.

					+1469				
				+1249		+1689			
Case OOO:			−1279				−1389		
		−2579						−1358	
		−2567	+2367		+2378		+3478	−3458	
					+1469				
		+1689						+1249	
Case OIO:		−1389		−3567		−3457		−1279	
		−1358		−2356	5	−4578		−2579	
				+2358		+2578			
					−1379				
Case IOI:		−1347	+1247	+2478		+2368	+3689	−3679	
			4		−2468		6		
		+1345	−1458	+1468		+2469	−2569	+5679	
			−1247	−2478		−2368	−3689		
	+3457	−1457		4	+2468	6		−3569	+3567
			+1458	−1468		−2469	+2569		
Case III:		+1579						+1359	
					5				

4.3 NPM30: A Pinched Torus.

The triangles in this case are

124	127	136	139	147	158	159	168	235	238	246	258
269	279	347	348	357	369	456	459	489	567	678	789

NPM30 has the three singular vertices 1, 2, and 3, each of type $(3,5)$. Its automorphism group is the cyclic group C_6 of order 6 which is generated by $\pi =$

(123)(456789). NPM30 is depicted in Figure 4. In this figure π appears as a glide reflection, π^2 is a pure translation and π^3 is a pure reflection.

There are many admissible oriented matroids all having trivial symmetry, but none with non-trivial symmetry. Hence we can expect all 8 topological types.

The tetrahedral partitions are as follows:

Case OOO:

			−1369	
	+2679	−6789	−1689	+1589
+1247	−2467			
	−4567	+3457	−3458	−4589
			−2358	

Case OOI:

	+2679	−6789	−3689		−1368	
+1247	−2467		+3489	3	+1358	−2358
	−4567	+3457	−3459		−1359	

Case OIO:

	−1568	+5678	+2578		−2357	
−1369	+1569		−2789	2	+2347	+1247
	−4569	−2469	+2489		−2348	

Case IOO:

	−3478	−4789	+1479		−1279	
−2358	+3578		−1459	1	+1269	−1369
	+5678	−1568	+1456		+1246	

Case III:

	+2357						
			2				
		+1238	−1258	−1245	−2456	+2567	−2679
			1	+1459			
		−1237	−1347	+1349	−3489	+3689	+6789
			3				
+1368							

The other three cases have no tetrahedral partition.

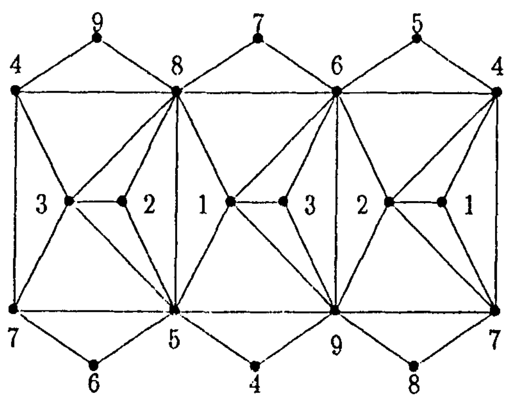

Figure 4: NPM30, a pinched torus

4.4 NPM01: A Pinched Manifold of Genus 2.

The triangles in this case are

124 235 346 457 568 167 278 138 125 236 347 458 156 267 378 148
139 249 359 469 579 689 179 289

This is a pinched manifold of genus 2 with one singular vertex, namely 9 which is of type (4,4). The automorphism group is C_8 generated by (12345678). In the terminology of block designs it is a 1-rotational design, see [18]. The blocks are generated by the above permutation. The vertex 9 plays the role of ∞ in the definition of 1-rotational designs $(9+1=9)$. NPM01 is depicted in Figure 5. It is the union of two tori with hole glued together along their boundary octagon (12345678).

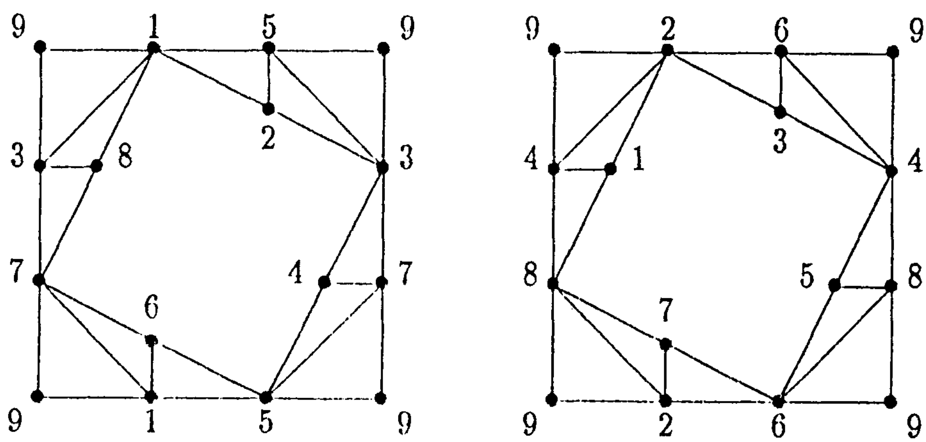

Figure 5: NPM01, a pinched manifold of genus 2

There is no oriented matroid with a symmetry of order 2 but there are many admissible oriented matroids without symmetry. We list two interesting admissible oriented matroids.

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-+++-+--+-+-+-+++-+-++-+--+-+-+---+-+-+-++-+--+---+-+-+++-++-

++++-+-+++-+++-++-+---+++-+++-++-+--+++--+++-+-+++-++-++-+++-++
-+++-+--+++-+-+++-+-++-+--+++-+---+-+-+-++-+--+---+---++--+++
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They differ by only 5 places, and they both share the same tetrahedral partition. Yet, as we will see in the next section, the first of them is not realizable while the second is. The genus 2 structure of this NPM can be easily seen from the arrangement of the tetrahedra in the partition.

Case O:	−1248	—	−2489	—	−4689		
	|				+4568		
	+1238	−2378	−2367	−3467	−4567		
	|				+1567		
	−1235	—	+1359	—	+1579		

For the second topological type (I) we picked a proper oriented matroid and it yielded the following tetrahedral partition.

Case I:	−1248	—	−2489	—	−4589	—	
	|				+4579		|
	+1238	−2378	−2367	−3467	−4679	9	−5689
	|				+1679		|
	−1235	—	+1359	—	+1569	—	

4.5 NPM45: Three 2-Spheres.

The triangles in this case are

				123	124	134	234					
156	167	178	189	159			357	379	369	368	358	
256	267	278	289	259			457	479	469	468	458	

This orientable neighborly pseudomanifold is connected, but not strongly connected : it is composed of three 2-spheres. The first 4 triangles form a tetrahedron; the next 10 triangles as well as the last 10 triangles form each a bi-pyramide over a pentagon. Each of these pentagons is the pentagram within the other pentagon, see Figure 6. Each of the vertices $1, 2, 3, 4$ is singular of type $(3, 5)$, and each of the other five vertices is singular of type $(4, 4)$.

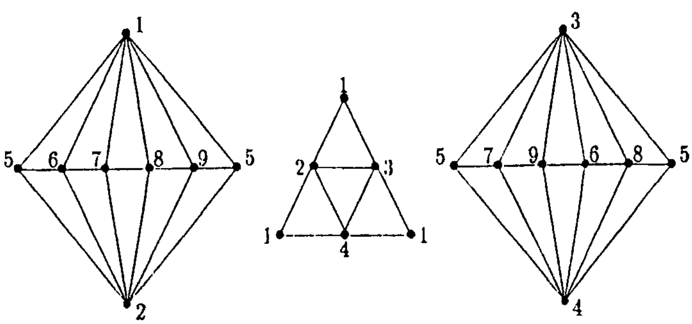

Figure 6: NPM45, three 2-spheres

This complex is highly symmetric. Its automorphism group is of order 80, generated by (12), (34), (56789), (13)(24)(6897).

NPM45 is the only of our five block designs, which has a subdesign. In [24] it is shown, that NPM45 is derived from $S_2(2,3,4)$ which is the boundary of the tetrahedron. As we shall see later in Section 5, a symmetry of order greater than 3 can not be realizable. There were altogether 128 oriented matroids which were reduced to only 18 essentially different cases. Two of these have a symmetry of order 4, one has a symmetry of order 2, and the remaining 15 have no symmetries. We found that in all 18 cases the convex hull (note that this is defined for oriented matroids as well) is a tetrahedron with vertices $1,2,3,4$. Note that in this case the distinction between I- and O-types mentioned at the end of Section 2 is meaningless. We picked an oriented matroid with the symmetry (12)(34)(5)(69)(78). As we will see in the next section, a tetrahedral partition is not needed here.

5. Neighborly Polyhedra

Our problem now is to realize the oriented matroids mentioned in Section 4, under the additional symmetry assumptions. This NP-hard problem of deciding the realizability has been described and discussed eg in [11] and in the literature cited there, see also [22]. For our purposes we have used refined methods. As the final results can be easily checked directly, we list the final results only.

In one case, in order to show that a certain symmetry can not be realized, we include a corresponding non-realizability proof.

5.1 NPM21: A Single Neighborly Polyhedron.

5.1.1 Theorem: *The neighborly pseudomanifold NPM21 described in Section 4 is realizable in R^3 with a two-fold symmetry.*

The coordinates are given by the matrix No. 1 in Section 6. The topological type of the singular vertices (see end of Section 2) is OIO, ie 2 and 8 are "outside" and 5 is "inside".

5.2 NPM03: Four Neighborly Polyhedra.

5.2.1 Theorem: *There are four type-different neighborly polyhedra with symmetry of order 2 and with boundary complex NPM03.*

The realizability of NPM03 in R^3 follows already from the observation that the mapping induced by (1)(2679845)(3) carries NPM03 to a subcomplex of the boundary complex of the neighborly 4-polytope with 9 vertices, denoted N^9_{20} in the list of all the neighborly 4-polytopes with 9 vertices in [3].

Any Schlegel diagram of N^9_{20} will yield a geometrical embedding of NPM03 in R^3, in which the convex hull of the neighborly polyhedron P obtained this way is a tetrahedron, see [16], Sec.3.3. By a proper choice of the facet of N^9_{20} to serve as a base for the Schlegel diagram, we can get $0, 1$, or 2 of the three singular vertices to be extreme vertices of P, that is, vertices of conv P.

It is interesting to note that none of the other NPM's is embeddable in the 2-skeleton of any convex 4-polytope with 9 vertices, and that among all the 23 neighborly 4-polytopes with 9 vertices, see [3], NPM03 is embeddable only in N^9_{20}.

Considering the nonpolytopal neighborly 3-manifolds we found that four of the NPM's are embeddable in neighborly 3-manifolds and neighborly 3-spheres, presented in [4]:

NPM21 is embeddable in N^9_{44} by (146957)(238), in N^9_{48} by (1746352)(8)(9) and in N^9_{51} by (1)(2)(368974)(5).

NPM03 is embeddable in N^9_{46} by (139426)(587), in N^9_{49} by (129784356), in N^9_{51} by (1)(268374)(5).

NPM30 is embeddable in N^9_{47} by (1)(258974)(36), in N^9_{50} by (142578936), in N^9_{51} by (1)(25643789).

NPM01 is embeddable in N^9_{42} by (127345689), in N^9_{50} by (1)(26473)(589).

NPM45 is not embeddable at all.

However, by this method we do not find symmetric realizations. When we look for symmetric realizations, we have :

5.2.2 THEOREM: *There is no geometrical realization for NPM03 with a symmetry of order 3.*

We consider the *final polynomial* used in the following proof to be of interest in its own right. Its type (affine conditions are involved as well) is different from those considered eg in [10].

Proof: According to 4.2 we have to show that the 3 neighborly matroid pseudomanifolds with the symmetry π mentioned there are not realizable. It is enough to show that there exists a final polynomial p for these oriented matroids with the property : $0 = p > 0$.

$$p = \underbrace{[1237]}_{>0}\underbrace{[1456]}_{<0}\underbrace{[4579]}_{<0} + \underbrace{[1246]}_{<0}\underbrace{[1278]}_{<0}\underbrace{[4567]}_{>0}$$
$$- \underbrace{[1246]}_{<0}\underbrace{[1279]}_{>0}\underbrace{[4567]}_{>0} - \underbrace{[1256]}_{>0}\underbrace{[1278]}_{<0}\underbrace{[4567]}_{>0} > 0.$$

We define the following bracket polynomial:

$$\{12 - fgh\} \ := \ [12fg] + [12gh] - [12fh] \ = \ 0.$$

This polynomial is zero as a result of the affine condition in rank 2. In order to see this, we delete the vertices 1 and 2. The points f, g and h are on a line. The property $[fg] + [gh] - [fh] = 0$ holds.
With the Grassmann-Plücker syzygy $\{45|1679\} := [4516][4579] - [4517][4569] + [4519][4567] = [1456][4579] - [1457][4567] + [1459][4567] = 0$ and because of the symmetry, we can transform the first summand. And we get the following form of p:

$$p =(\underbrace{[1237][1457]}_{1} - \underbrace{[1237][1459]}_{2} + \underbrace{[1246][1278]}_{3}$$
$$- \underbrace{[1246][1279]}_{4} - \underbrace{[1256][1278]}_{5})\,[4567]$$

Now we can use the following Grassmann-Plücker syzygies to transform p.

$$\{17|2345\} := \underbrace{[1237][1457]}_{1} + [1247][1248] - [1257][1268] = 0$$
$$\{19|2345\} := \underbrace{[1237][1459]}_{2} + [1249][1247] - [1259][1267] = 0$$
$$\{12|4678\} := \underbrace{[1246][1278]}_{3} - [1247][1268] + [1248][1267] = 0$$
$$\{12|4679\} := \underbrace{[1246][1279]}_{4} - [1247][1269] + [1249][1267] = 0$$
$$\{12|5678\} := \underbrace{[1256][1278]}_{5} - [1257][1268] + [1258][1267] = 0$$

$$\begin{aligned} p \ = \ (-\ & [1247][1248] + [1257][1268] + [1249][1247] - [1259][1267] \\ +\ & [1247][1268] - [1248][1267] - [1247][1269] + [1249][1267] \\ -\ & [1257][1268] + [1258][1267]\,)\,[4567] \\ = \ (\ \ & [1247] \quad (-\{12-489\} + \{12-689\}) \\ +\ & [1267] \quad (-\{12-489\} + \{12-589\})\,)[4567] = 0 \end{aligned}$$

So we have $p = [4567][1247](\{12-689\} - \{12-489\}) + [4567][1267](\{12-589\} - \{12-489\}) = 0$.

Thus our theorem is proved.

Thus the symmetry σ in Theorem 5.2.1 is best possible. The matrices No.2, ..., No.5 in Section 6. provide coordinates for the 4 polyhedra realizing NPM03 with the symmetry σ.

We show (Figure 7) parallel projections of one half of the second polyhedron P (Case OIO). This example was chosen, because it seemed to be in a sense the easiest to understand. We have chosen an exploded view as well. That is, in order to improve the visualization of the structure, the tetrahedra are moved apart (as if a small explosion in some interior point of the body). The rotational symmetry with a vertical axis passing through 5 provides the key for a full comprehension.

5.3 NPM30: Eight Neighborly Polyhedra.

5.3.1 Theorem: *There are 8 type-different neighborly polyhedra with boundary complex NPM30.*

The affine coordinates are listed as matrices No.6, ..., No.13 in Section 6.

5.4 NPM01: Two Neighborly Polyhedra.

5.4.1 Theorem: *There are 2 type-different neighborly polyhedra with boundary complex NPM01.*

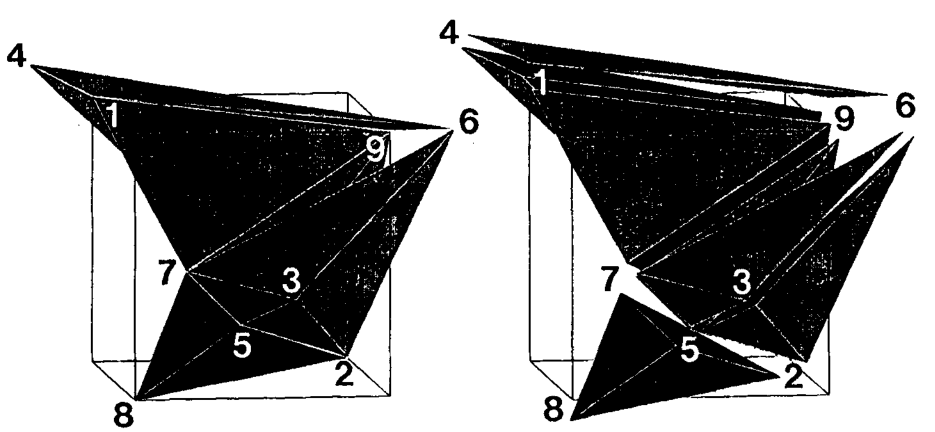

Figure 7: NPM03, Case OIO

In other words, the two topological cases described in Section 4.4 are realizable. To be more precise, the first oriented matroid described there is not realizable. There exists a bi-quadratic final-polynomial, see [10] for this notion. The second one and the third, which was not described in detail, are realizable with the coordinates in matrices No.14 (O) and No.15 (I) in Section 6.

5.5 NPM45: A Single Neighborly Polyhedron.

5.5.1 Theorem: *There is a neighborly polyhedron with a symmetry of order 2 whose boundary complex is NPM45.*

The symmetry is (12)(34)(5)(69)(78) and the coordinates are given in the matrix No.16 in Section 6.

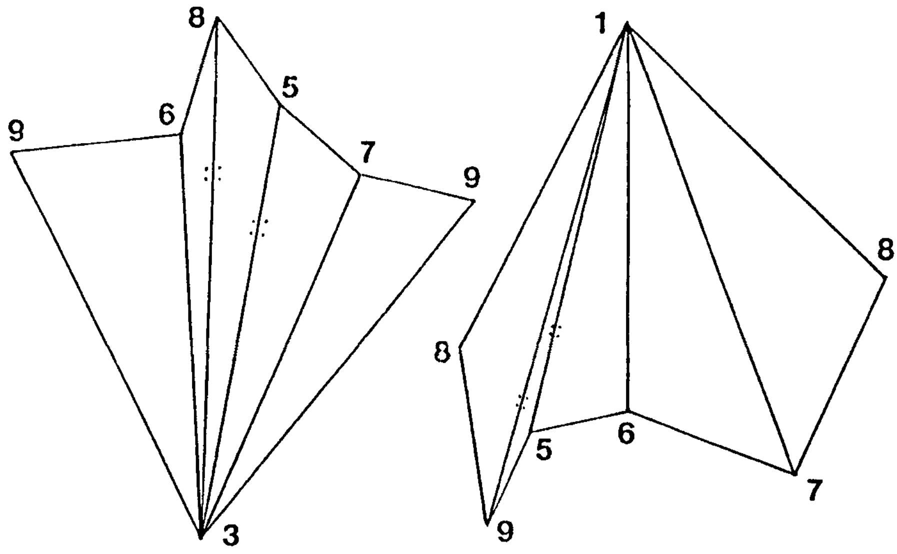

Figure 8: The unfolded boundary complex of the holes inside the neighborly polyhedron realizing NPM45

The convex hull is a tetrahedron T with the vertices 1,2,3 and 4. This polyhedron differs essentially from all the previous ones. It consists of the tetrahedron T with two interlaced holes inside it. Each hole has the shape of a bi-pyramide over a non-planar pentagon (see Figure 6) and resembles a bumerang. The two *bumerangs* touch each other in the five vertices 5,6,7,8,9. The boundary complex of each of these bi-pyramides consists of two congruent parts, each part consisting of the five triangles sharing an apex of the bi-pyramide. Figure 8 depicts the unfolded boundary complexes of these two bi-pyramides in correct scale. With it a paper model for the hole in this polyhedron becomes clear. To facilitate a proper folding of the paper, the edges at which the angle interior to the bi-pyramidal hole is greater than π are indicated by dots.

We have seen (Sec.4.5) that a neighborly matroid pseudomanifold of NPM45 has symmetries of order 4. None of these symmetries can be realized geometrically, for the following reason. Each of these symmetries has at least two fixed points and therefore its geometric counterpart is a rotation or reflection. This, in turn, implies that certain four vertices form the vertices of a planar convex quadrangle, which contradicts the neighborliness. Thus the realization of NpM45 described above is of maximal symmetry.

6. Coordinate Matrices

We list the coordinate matrices of all polyhedra in the following ordering : NPM21; NPM03 (OOO, OIO, IOI, III); NPM30 $(OOO, OOI, OIO, IOO, OII, IOI, IIO, III)$; NPM01 (O, I); NPM45.

$$\begin{pmatrix} 5 & -5 & 5 \\ 3.4 & -6.5 & 2 \\ -5 & -5 & -5 \\ 1 & -3.2 & 2.1 \\ 0 & 0 & 1.3 \\ -1 & 3.2 & 2.1 \\ 5 & 5 & -5 \\ -3.4 & 6.5 & 2 \\ -5 & 5 & 5 \end{pmatrix} \begin{pmatrix} -5 & 5 & -5 \\ 5 & 5 & 5 \\ -3.06452 & -4.03226 & 4.67742 \\ -5 & 35 & -15 \\ 0 & 0 & 2.5 \\ 5 & -35 & -15 \\ 3.06452 & 4.03226 & 4.67742 \\ -5 & -5 & 5 \\ 5 & -5 & -5 \end{pmatrix}$$

$$\begin{pmatrix} -5 & 5 & 5 \\ 5 & 5 & -5 \\ 1.52 & -3.7 & -1.52 \\ -6.88 & 8.75 & 5.63 \\ 0 & 0 & -3 \\ 6.88 & -8.75 & 5.63 \\ -1.52 & 3.7 & -1.52 \\ -5 & -5 & -5 \\ 5 & -5 & 5 \end{pmatrix} \begin{pmatrix} 2.66 & 0 & -1 \\ -0.75 & 0 & -1 \\ 1.41 & 54.36 & 79.88 \\ 2.1 & -0.8 & -1.28 \\ -68.33 & -68.33 & 0 \\ -0.8 & 2.1 & 1.28 \\ 54.36 & 1.41 & -79.88 \\ 0 & -0.75 & 1 \\ 0 & 2.66 & 1 \end{pmatrix}$$

$$\begin{pmatrix} -5 & -5 & 5 \\ 0.57 & 0.06 & -4.36 \\ -0.25 & 0.16 & -4.15 \\ 5 & -5 & -5 \\ 0 & 0 & -4.05 \\ -5 & 5 & -5 \\ 0.25 & -0.16 & -4.15 \\ -0.57 & -0.06 & -4.36 \\ 5 & 5 & 5 \end{pmatrix} \begin{pmatrix} -5 & 5 & -5 \\ 5 & 5 & 5 \\ -5 & -5 & 5 \\ -2.79 & -0.25 & -7.49 \\ -3.31 & -1.38 & -4.43 \\ -2.85 & 1.25 & -3.36 \\ 5 & -5 & -5 \\ -4.2844 & -0.11 & -3.6338 \\ -4.3036 & -0.08 & -3.6227 \end{pmatrix}$$

$$\begin{pmatrix} -5 & 5 & -5 \\ 5 & 5 & 5 \\ -5 & -5 & 5 \\ -8.33 & -5 & 5 \\ -5.91 & -4.09 & 5.91 \\ -35 & 5 & -15 \\ -8.85 & -5.25 & 4.84 \\ 5 & -5 & -5 \\ -8.82 & -4.08 & 3.32 \end{pmatrix} \begin{pmatrix} -1.3 & -0.96 & -1.75 \\ 1.33 & 2.07 & -0.12 \\ 0.74 & 0 & -0.89 \\ -2.41 & 0 & -0.89 \\ 0 & -2.62 & 1.12 \\ 0 & 0.86 & 1.12 \\ 2.54 & 2.55 & 0.03 \\ 0.71 & 0.69 & 0.5 \\ -0.54 & 0.58 & 0.63 \end{pmatrix}$$

$$\begin{pmatrix} 2.215 & -0.149 & 2.903 \\ -5 & 5 & 5 \\ 1.636 & -1.718 & 4.954 \\ 2.48 & -0.139 & 2.619 \\ 2.212 & -0.155 & 2.896 \\ 1.595 & 0.345 & 3.023 \\ 5 & 5 & -5 \\ -5 & -5 & -5 \\ 5 & -5 & 5 \end{pmatrix} \begin{pmatrix} -0.97 & 0 & -2.71 \\ -2.22 & -2.64 & -0.81 \\ -0.19 & 0.28 & 0.77 \\ 8.59 & 0 & -2.71 \\ 0 & -0.02 & 0.37 \\ -1.09 & -1.33 & -0.87 \\ -3.87 & -64.35 & -1.14 \\ -0.71 & -1.6 & 1.13 \\ 0 & 64.47 & 0.37 \end{pmatrix}$$

$$\begin{pmatrix} -0.06 & 0 & -0.12 \\ 17.65 & 0 & -0.12 \\ -0.42 & -3.42 & -7.31 \\ 6.58 & -10.69 & 7.82 \\ 0 & -15.62 & 8.21 \\ 0 & 4.38 & 8.21 \\ -3.13 & 2.31 & -14.87 \\ -0.09 & -13.21 & 6.88 \\ -0.49 & 0.09 & -0.41 \end{pmatrix} \begin{pmatrix} 2.6923 & 2.3077 & 3.8462 \\ 3.2051 & 3.0769 & 3.8462 \\ 2.6798 & 2.3055 & 3.8277 \\ 3.0986 & 2.9577 & 3.8732 \\ 2.7164 & 2.3468 & 3.8474 \\ -5 & -5 & 5 \\ 5 & 5 & 5 \\ 5 & -5 & -5 \\ -5 & 5 & -5 \end{pmatrix}$$

$$\begin{pmatrix} 5 & 5 & -5 \\ -5 & -5 & -5 \\ 2.75 & -0.17 & 2.07 \\ -5 & 5 & 5 \\ -0.9 & 0.2 & -0.9 \\ -0.65 & 0.45 & -0.5 \\ 5 & -5 & 5 \\ 1.97 & -0.06 & 1.43 \\ -1.45 & 1.45 & 0.7 \end{pmatrix} \begin{pmatrix} 5 & 5 & -5 \\ 3.82 & -0.61 & -0.58 \\ -5 & -5 & -5 \\ 3.87 & 0.05 & 0.43 \\ 4.03 & 0.88 & -0.03 \\ 3.97 & -0.69 & -0.33 \\ 5 & -5 & 5 \\ 3.92 & -0.67 & -0.58 \\ -5 & 5 & 5 \end{pmatrix}$$

$$\begin{pmatrix} -193.55 & 0 & -4.56 \\ 0 & 1.00 & 0.22 \\ 42.43 & 188.85 & -4.42 \\ 0.57 & 1.54 & -0.88 \\ -79.76 & 3.64 & -4.39 \\ 0 & -1.04 & 0.22 \\ 193.52 & -2.20 & 5.12 \\ -4.40 & -0.94 & 0.71 \\ 0.11 & 0 & -4.56 \end{pmatrix} \begin{pmatrix} 5 & 5 & -5 \\ -5 & -5 & -5 \\ -5 & 5 & 5 \\ 5 & -5 & 5 \\ 0 & 0 & -0.4 \\ 1 & 1.5 & 0.6 \\ -2.15 & 0.2 & -0.7 \\ 2.15 & -0.2 & -0.7 \\ -1 & -1.5 & 0.6 \end{pmatrix}$$

ACKNOWLEDGEMENT: We wish to thank Netanel Altshuler for the programming involved in embedding the NPM's in convex 4-polytopes as mentioned in Section 5.2. A new program normalizing oriented matroids implemented by António Guedes de Oliveira was of help for reducing the number of oriented matroids in the NPM45 case, we wish to thank him as well. We thank Gabriel Zachmann for the implementation of a computer graphics program system. Finally, we wish to express our gratitude to the referee for many helpful improvements, especially those concerning the connection to block designs.

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